Seshadri constants and very ample divisors on algebraic surfaces

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Abstract: A broadly applicable geometric approach for constructing nef divisors on blow ups of algebraic surfaces at n general points is given; it works for all surfaces in all characteristics for any n. This construction is used to obtain substantial improvements for currently known lower bounds for n point Seshadri constants. Remarks are included about a range of applications to classical problems involving linear systems on $\mathbf{P}^2$.

I. Introduction

This paper presents a broadly applicable geometric approach to building nef divisors on surfaces. Our main application is to obtaining bounds on multipoint Seshadri constants for n general points on surfaces X. What we find is that, for n sufficiently large, all of the main results for $X = \mathbf{P}^2$ hold for surfaces generally.

To begin, let X be an algebraic surface (by which we will always mean a reduced, irreducible, normal projective variety of dimension 2, over an algebraically closed field of arbitrary characteristic). Let L be a nef divisor on X, let $l = L^2$, and let $p_1, \ldots, p_n$ be distinct points of X. Seshadri constants were introduced in [De]; more generally, multiple point versions have been studied in [Bau], [Bi], [Ku1], [Xu2], [S] and [ST]. To recall, the multiple point Seshadri constant $\epsilon(L, p_1, \ldots, p_n)$ is defined to be the supremum of all rational numbers ε such that $\pi^*L - \varepsilon(E_1 + \cdots + E_n)$ is a nef $\mathbf{Q}$-divisor, where $\pi : Y \to X$ is the morphism blowing up the points p_i, $1 \leq i \leq n$, and E_i is the exceptional divisor corresponding to p_i. We will often be concerned with finding lower bounds for $\epsilon(L, p_1, \ldots, p_n)$ which hold on an open set in X^n. Thus, given a lower bound c, it will be convenient to write $\epsilon(L, n) \geq c$ to mean that $\epsilon(L, p_1, \ldots, p_n) \geq c$ holds on an open set of n-tuples of points p_i of X.

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It is not hard to see that $\epsilon(L, p_1, \ldots, p_n) \leq \sqrt{l/n}$ (see Section III), and, as remarked in [Xu2], it follows over $\mathbf{C}$ from [EL] that $\epsilon(L, p_1, \ldots, p_n) \geq 1$ for sufficiently general points p_i, if $l > n$. Another lower bound for $\epsilon(L, p_1, \ldots, p_n)$ for sufficiently general points p_i over $\mathbf{C}$ follows from the main result of [Ku1], but (in dimension 2) this lower bound is never more than $\sqrt{l/n}\sqrt{1-1/n}$. Our results are of interest mainly when $l < n$: for any given very ample divisor L, our main result, Theorem I.1, obtains (in view of Proposition I.2) better bounds than $\sqrt{l/n}\sqrt{1-1/n}$ for almost all n sufficiently large.

There are very few cases for which the value of $\epsilon(L, p_1, \ldots, p_n)$ is known for n general or generic points p_i. One important case, which has seen a great deal of attention beginning with Nagata's work on Hilbert's 14th Problem, is when L is very ample and $l = 1$, which forces X to be $\mathbf{P}^2$, but even here, $\epsilon(L, n)$ is known only when $n < 9$ (see Remark III.7) or when n is a square. In fact, Nagata's conjecture [N1], that (in different terminology) $\epsilon(L, p_1, \ldots, p_n) = 1/\sqrt{n}$ should hold for $n > 9$ generic points p_i of $\mathbf{P}^2$ even when n is not a square, is still open. (When n is a square, it is easy to check that $\epsilon(L, n) \geq 1/\sqrt{n} - \varepsilon$ holds for n *general* points for any positive rational ε; however, our generalization of this in Theorem I.1 to any surface seems to be both new and nontrivial.)

Some of the best results obtained so far for the case that $l = 1$ and $X = \mathbf{P}^2$ over $\mathbf{C}$ are due to Biran [Bi], who uses a powerful procedure for building nef divisors. Although ad hoc applications of this procedure can yield impressive results in particular cases (such as the calculation $\epsilon(L, p_1, \ldots, p_{19}) \geq 39/170$ in section 5 of [Bi], whereas our Theorem I.1 gives only 39/171), obtaining general results by this procedure seems to require carefully constructed values of n. For example, given positive integers a and i, Theorem 2.1A of [Bi] gives bounds if $n = a^2i^2 \pm 2i$, or if $n = a^2i^2 + i$ and $ai \geq 3$. But in these cases there are certain positive integer solutions to $r^2 - d^2n = 1$: for $n = a^2i^2 \pm 2i$, take $r = a^2i \pm 1$ and $d = a$, and for $n = a^2i^2 + i$, take $r = 2a^2i + 1$ and $d = 2a$; either way Biran's bound is $\sqrt{1/n}\sqrt{1-1/r^2}$. Applying Proposition I.2(b), we recover as a special case of Theorem I.1 these same bounds in those cases with $r \leq n$, and we obtain even better bounds via Proposition I.3 when $n = a^2i^2 \pm 2i$ and $a = 1$ (and hence $n+1$ is a square). For the cases when $i = 1$ and either $n = a^2i^2 - 2i$ or $n = a^2i^2 + i$, we have $r > n$ so Proposition I.2(b) does not apply, but (except in the case that $n = a^2 + 1$ and either the characteristic is 2 or a is a power of 2) Proposition I.3 recovers Biran's bound via a refined application of our underlying approach; see Section IV. Similarly, if $n = a^2i^2 - i$, which [Bi] does not treat (except in special cases when n can also be written in the form $n = a'^2i'^2 \pm 2i'$), we can take $d = 2a$ and $r = 2a^2i - 1$ and again obtain the bound $\sqrt{1/n}\sqrt{1-1/r^2}$, as long as $i > 2$. (For bounds when $i \leq 2$, and more generally when $n+2$ or $n+1$ is a square, see Proposition I.3.)

Another result over $\mathbf{C}$ for $X = \mathbf{P}^2$ and $l = 1$ that should be mentioned is that $\epsilon(L, n) \geq 1/\sqrt{n+1}$ for $n \geq 10$ general points [ST]. Apart from cases which follow from [H2] (which this paper generalizes) and from those of [Bi] just mentioned, this seems to have been the best estimate known up to now. However, Theorem I.1 with Proposition I.2(c) (or Proposition I.3 if $n\pm 1$ is a square) is better in all cases. Moreover, our approach applies to all surfaces in all characteristics. [Very recently, by a very elegant argument for $X = \mathbf{P}^2$ over $\mathbf{C}$, Szemberg [S] obtained a bound of the form $\epsilon(L, n) \geq (1/\sqrt{n})\sqrt{1-1/(an)}$, where a currently can be as large as about 5. But for n sufficiently large, the bound of Theorem I.1 is, by Proposition I.2(b)(iii), better except for a small fraction of cases. Nevertheless,

there are some special values of n of particular interest, including certain small values of n and when $n = s^2 - 1$ where $s - 1$ is a power of 2, for which the bound of [S] is the best one we know.]

In short, Seshadri constants are difficult to compute and in general remain unknown, but they are closely connected to classical problems involving linear systems and thus are of substantial interest. In this paper, using more broadly applicable geometric methods than have been typical of work on this problem, we give a characteristic free approach to estimating Seshadri constants that nonetheless gives comprehensive improvements to currently known lower bounds.

In preparation for stating our main result, let l and n be positive integers and define the sets

$$S_1(n,l) = \left\{ \frac{r}{nd} \;\middle|\; 1 \leq r \leq n, \quad 1 \leq d, \quad \frac{r}{d} \leq \sqrt{nl} \right\}$$

and

$$S_2(n,l) = \left\{ \frac{dl}{r} \;\middle|\; 1 \leq r \leq n, \quad 1 \leq d, \quad \frac{r}{d} \geq \sqrt{nl} \right\}$$

of integer ratios. Now define $S(n,l) = S_1(n,l) \cup S_2(n,l)$ and $\varepsilon_{n,l} = \max(S(n,l))$. With a view to the important special case that L is a line in $X = \mathbf{P}^2$, we will write ε_n for $\varepsilon_{n,1}$.

We now have the following result (proved in Section III as Theorem III.1):

Theorem I.1: *Let $l = L^2$, where L is a very ample divisor on an algebraic surface X. Then $\sqrt{l/n} \geq \epsilon(L,n)$, and in addition, we have $\epsilon(L,n) \geq \varepsilon_{n,l}$ unless $l \leq n$ and nl is a square, in which case $\sqrt{l/n} = \varepsilon_{n,l}$ and $\epsilon(L,n) \geq \sqrt{l/n} - \varepsilon$ for every positive rational ε.*

(The somewhat awkward statement in case nl is a square is related to there possibly being no *open* set of points such that $\epsilon(L,n) = \varepsilon_{n,l}$ in that case.)

Note that $\varepsilon_{n,l}$ is just the maximum element in the finite set

$$\left\{ \frac{\lfloor d\sqrt{nl} \rfloor}{dn} \;\middle|\; 1 \leq d \leq \sqrt{\frac{n}{l}} \right\} \cup \left\{ \frac{1}{\lceil \sqrt{\frac{n}{l}} \rceil} \right\} \cup \left\{ \frac{dl}{\lceil d\sqrt{nl} \rceil} \;\middle|\; 1 \leq d \leq \sqrt{\frac{n}{l}} \right\}.$$

Thus for any given n it is not hard to compute $\varepsilon_{n,l}$ exactly, even though it is not easy to give an explicit formula. As an alternative, we give some comparisons and in addition determine $\varepsilon_{n,l}$ explicitly in some cases (the proof of Proposition I.2 is in Section III):

Proposition I.2: *Let l, s and n be positive integers.*

(a) If $l \geq n$, then $\varepsilon_{n,l} = 1$.

(b) Say $l < n$, let d and $r \leq n$ be positive integers and put $\delta = r^2 - nld^2$.

(i) If nl is a square, then $\varepsilon_{n,l} = \sqrt{l/n}$.

(ii) We have

$$\varepsilon_{n,l} \geq \sqrt{\frac{l}{n}}\sqrt{1 - \frac{\delta}{r^2}} \text{ if } \delta \geq 0 \text{ and } \varepsilon_{n,l} \geq \sqrt{\frac{l}{n}}\sqrt{1 + \frac{\delta}{nld^2}} \text{ if } \delta \leq 0,$$

with equality if $\delta = \pm 1$.

(iii) Moreover,

$$\varepsilon_{n,l} > \sqrt{\frac{l}{n}}\sqrt{1 - \frac{1}{n}} \qquad (*)$$

holds for at least half of the values of l from $(n-1)/2$ to $n-1$ as long as $n>2$. Alternatively, given any positive integer a and $s>2$, the fraction of the number of values of n in the range $s^2l \le n < (s+1)^2l$ for which

$$\varepsilon_{n,l} > \sqrt{\frac{l}{n}}\sqrt{1-\frac{1}{an}} \qquad (**)$$

fails to hold is at most $((2a^2-a+8)l+3)/(l(2s+1))$ if $a>2$, $(14l+3)/(l(2s+1))$ if $a=2$, and $(4l+2)/(l(2s+1))$ if $a=1$, and thus goes to 0 as s increases.

(c) *If $n\pm 1$ is not a square, we have*

$$\varepsilon_n > \frac{1}{\sqrt{n+1}} > \sqrt{\frac{1}{n}}\sqrt{1-\frac{1}{n}}.$$

For more explicit estimates of $\varepsilon_{n,l}$ and ε_n, see Corollary III.2 and Corollary III.5. Also, for a given n, we note that the first statement of Proposition I.2(b)(iii) significantly understates the number of l from 1 to n for which $(*)$ holds, which often is $3n/4$ or more; see Remark III.8.

Regarding Proposition I.2(c), it is especially difficult to improve on previously known bounds when n is close to a square. If $n\pm 1$ or $n+2$ is a square, we can improve on ε_n using a refinement of our basic approach. We obtain the following result, proved in Section IV, which in all cases is better than $1/\sqrt{n+1}$, and recovers Biran's bound if either $n+2$ is a square or (in certain cases) if $n-1$ is a square, and improves on Biran's bound if $n+1$ is a square:

Proposition I.3: *Let n be a positive integer, with L a line in $X=\mathbf{P}^2$; we have:*

- *if $9\le n+2$ is a square then $\epsilon(L,n) \ge \sqrt{\frac{1}{n}}\sqrt{1-\frac{1}{(n+1)^2}}$;*
- *if $9\le n+1$ is a square, then $\epsilon(L,n) \ge \sqrt{\frac{1}{n}}\sqrt{1-\frac{n-1}{n(\sqrt{n+1}+1)^2}}$, and (unless $\sqrt{n+1}-1$ is a power of 2 or the characteristic is 2) $\epsilon(L,n) > \sqrt{\frac{1}{n}}\sqrt{1-\frac{1}{(\sqrt{n+1}-1)n}}$;*
- *if $9\le n-1$ is a square, then $\epsilon(L,n) \ge \sqrt{\frac{1}{n}}\sqrt{1-\frac{n-1}{(n+\sqrt{n-1})^2}}$, and (unless $n-1$ is a power of 4 or the characteristic is 2) $\epsilon(L,n) \ge \sqrt{\frac{1}{n}}\sqrt{1-\frac{1}{(2n-1)^2}}$.*

We include two corollaries that may be of interest. For any nl not a square, there are infinitely many solutions (r,d) to $r^2-d^2nl=1$. Unfortunately, if r is too big we cannot apply Proposition I.2(b)(ii) to obtain a bound. By the next result (see Section III for the proof), such solutions need not entirely go wasted:

Corollary I.4: *Let L be a very ample divisor on a surface X with $l=L^2$ and consider positive integers n, r and $d=ab$. If $r^2-nld^2=1$ and $a>b\sqrt{l/n}$, then $\pi^*L-(bl/r)(E_1+\cdots+E_{a^2n})$ is a nef $\mathbf{Q}$-divisor, where $\pi: Y\to X$ is the birational morphism obtained by blowing up a^2n general points p_i, E_i being the exceptional divisor corresponding to p_i; in particular, we have $\epsilon(L,a^2n)\ge bl/r$.*

The preceding result is suggestive of the procedure of [Bi] on $X = \mathbf{P}^2$ over $\mathbf{C}$, which, for example, can easily be used to show that $at\pi^*L - m(E_1 + \cdots + E_{a^2n})$ is nef if $t\pi^*L - m(E_1 + \cdots + E_n)$ is. Similarly, the next corollary (proved in Section III) generalizes two additional facts known on $X = \mathbf{P}^2$ over $\mathbf{C}$: that the divisor $H = t\pi^*L - (E_1 + \cdots + E_n)$ is ample if $H^2 > 0$ (see [Xu1] or [Ku2]) and $H = t\pi^*L - 2(E_1 + \cdots + E_n)$ is nef if $H^2 \geq 0$ (see Theorem 2.1.B of [Bi]).

Corollary I.5: *Let L be a very ample divisor on a surface X with $l = L^2$ and consider positive integers $n > l$ and $d \leq \sqrt{n/l}$. Let $\pi : Y \to X$ be the birational morphism obtained by blowing up n general points p_i, E_i being the corresponding exceptional divisor. Then $H_r = r(\pi^*L) - dl(E_1 + \cdots + E_n)$ is a nef divisor for all integers $r > d\sqrt{nl}$, and an ample $\mathbf{Q}$-divisor for all rationals $r > \lceil d\sqrt{nl}\rceil$.*

Our approach uses an explicit construction in Section II of nef divisors on the blow up Y of X at the points p_i, with the points taken in special position. As a consequence of these nef divisors, we obtain various bounds in Section III. The construction of nef divisors given in Section II can be refined to sometimes obtain better bounds. Since doing this introduces some complications, we segregate this material to Section IV. In Section V we discuss additional applications of the existence of these nef divisors to various classical problems involving linear systems on $\mathbf{P}^2$. Analogous remarks could be made for surfaces more generally, but complications (such as irregular surfaces, and failure of vanishing theorems to hold in certain circumstances in positive characteristics) arise that would require special treatment. Thus we leave such remarks for the reader to work out in cases of his or her own interest.

II. Nef Divisors on Blow ups

The foundation for our results is a method for constructing nef divisors generalizing what is done in [H2]. To state our basic lemma, let X_0 be an algebraic surface. We will call a sequence $p_1, \cdots, p_n$ of points a *proximity sequence* if $p_1 \in X_0$ is smooth, p_2 is a point of the exceptional divisor of the blow up $\pi_1 : X_1 \to X_0$ at p_1, and, for $2 < i < n$, $\pi_{i-1} : X_{i-1} \to X_{i-2}$ is the blow up of X_{i-2} at p_{i-1} and p_i is a point of the exceptional divisor of π_{i-1}, but not a point of the proper transform of the exceptional divisor of π_{i-2}. Denote the composition $\pi_n \circ \cdots \circ \pi_1$ by $\pi : X_n \to X_0$ and denote by E_i the scheme theoretic fiber $(\pi_n \circ \cdots \circ \pi_{i-1})^{-1}(p_i)$. Thus E_i is a divisor on X_n, the total transform of p_i, and, for each $1 \leq i < n$, $[E_i - E_{i+1}]$ is the class of a reduced irreducible divisor.

Lemma II.1: *Let $\pi : Y \to X$ be the morphism obtained by blowing up general points $p_1, \cdots, p_n$ of an algebraic surface X, and let E_i be the exceptional divisor corresponding to each point p_i. Let $\pi' : Y' \to X$ be the morphism corresponding to a proximity sequence $p'_1, \cdots, p'_n$ for X, with E'_i being the exceptional divisor on Y' corresponding to p'_i. Suppose we are given a divisor L on X with $l = L^2$, and integers $1 \leq d$, $1 \leq r \leq n$, and $m_1 \geq \cdots \geq m_r \geq 0$ such that $[d(\pi'^*L) - m_1E'_1 - \cdots - m_rE'_r]$ is the class of an irreducible effective divisor C. Then $a_0d(\pi^*L) - a_1E_1 - \cdots - a_nE_n$ is a nef $\mathbf{Q}$-divisor on Y for any rational numbers a_i satisfying:*

- $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$;
- $a_0d^2l \geq a_1m_1 + \cdots + a_rm_r$;

- $a_0^2 d^2 l > a_1^2 + \cdots + a_n^2$;
- $(m_1 + \cdots + m_i)a_0 \geq a_1 + \cdots + a_i$ *for* $1 \leq i \leq r$; *and*
- $(m_1 + \cdots + m_r)a_0 \geq a_1 + \cdots + a_n$.

Proof: The point is to show $N = a_0 d(\pi'^* L) - a_1 E'_1 - \cdots - a_n E'_n$ is nef on Y'. Since by assumption $N^2 > 0$, the result then follows since being nef and of positive self-intersection is an open condition for flat families of line bundles. But the divisor class $[N]$ is nef on Y' because it is a nonnegative $\mathbf{Q}$-linear combination of classes of irreducible effective divisors, each of which N meets nonnegatively. In particular, the last two bulleted hypotheses guarantee that N is the sum of $a_0 C$ with various nonnegative multiples of $E_i - E_{i+1}$ for $1 \leq i < n$ and E_n. The first two bulleted hypotheses guarantee that N meets each summand nonnegatively. (It follows that $a_0^2 d^2 l \geq a_1^2 + \cdots + a_n^2$, i.e., $N^2 \geq 0$, is a consequence of the other hypotheses, and that $N^2 > 0$ is automatic unless: $a_0 d^2 l = a_1 m_1 + \cdots + a_r m_r$, and $((m_1 + \cdots + m_i)a_0 - (a_1 + \cdots + a_i))(a_i - a_{i+1}) = 0$ for $1 \leq i < r$, and $(a_r - a_i)a_i = 0$ for $i > r$, and $(m_1 + \cdots + m_r)a_0 = a_1 + \cdots + a_n$.) $\diamondsuit$

We now apply the preceding lemma in case L is a very ample divisor on X.

Lemma II.2: *Let* $\pi : Y \to X$ *be the morphism blowing up general points* $p_1, \ldots, p_n$ *of an algebraic surface* X*, let* $E_1, \cdots, E_n$ *be the exceptional divisors corresponding to the points* p_i*, let* L *be a very ample divisor on* X *and put* $l = L^2$ *and* $L' = \pi^* L$*. Given positive integers* $r \leq n$ *and* d*, and nonnegative rational numbers (not all 0)* $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$ *such that* $a_0 d^2 l \geq a_1 + \cdots + a_r$*,* $a_0^2 d^2 l > a_1^2 + \cdots + a_n^2$ *and* $r a_0 \geq a_1 + \cdots + a_n$*, then* $(a_0 d)L' - (a_1 E_1 + \cdots + a_n E_n)$ *is a nef* $\mathbf{Q}$*-divisor on* Y*.*

Proof: Since L is very ample, $|dL|$ has an irreducible member C' passing through and smooth at some smooth point $p'_1 \in X$. Take as our proximity sequence points of C' infinitely near to p'_1; i.e., blow up p'_1 and take p'_2 to be the point of the proper transform of C' infinitely near to p'_1. Similarly define p'_i for all $i \leq r$. Then extend to a proximity sequence $p'_1, \ldots, p'_n$ such that p'_{r+1} is *not* on the proper transform of C'. Blowing up the points of the sequence gives the morphism $\pi' : Y' \to X$, and $[d(\pi'^* L) - E'_1 - \cdots - E'_r]$ is the class of the proper transform of C', which is irreducible. It is now easy to check that the hypotheses of Lemma II.1 apply (take $m_i = 1$ for all i), giving the result.$\diamondsuit$

One application of Lemma II.2 is to provide nef divisors F which can be employed to test for effectivity: given a divisor H on X and integers b_i (we may as well assume $b_1 \geq \cdots \geq b_n \geq 0$), if $F \cdot (\pi^* H - b_1 E_1 - \cdots - b_n E_n) < 0$ for some nef F, then $|\pi^* H - b_1 E_1 - \cdots - b_n E_n|$ is empty. Given $\pi^* H - b_1 E_1 - \cdots - b_n E_n$, the optimal nef test divisor F provided by Lemma II.2 can be found by linear programming. (Keep in mind that we can always normalize so that $a_0 = 1$, and clearly one need consider only finitely many r and d.)

In order to avoid linear programming, the following corollary obtains some special cases of particular interest.

Corollary II.3: *Given* X*,* Y*,* l*,* n *and* L' *as in Lemma II.2, let* d *and* $r \leq n$ *be positive integers. Then we have the following cases.*

(a) *If* $r^2 > nd^2 l$*, then* $rL' - dl(E_1 + \cdots + E_n)$ *is nef.*
(b) *If* $r^2 < nd^2 l$*, then* $ndL' - r(E_1 + \cdots + E_n)$ *is nef.*
(c) *If* $r^2 = nd^2 l$*, then* $tL' - (E_1 + \cdots + E_n)$ *is nef for all rationals* $t > \sqrt{n/l}$*.*

Proof: Apply Lemma II.2 for various values of the a_i. For (a), take $a_0 = r/d$ and $a_1 = \cdots = a_n = ld$. For (b), take $a_0 = n$ and $a_i = r$, $i > 0$. For (c), take $a_0 > n/r$ and $a_i = 1$, $i > 0$. $\diamondsuit$

There are cases when one may want to construct nonuniform nef divisors. Here are some examples of such.

Corollary II.4: *Given X, Y, l, n and L' as in Lemma II.2, let d and $r \leq n$ be positive integers. Then we have the following cases.*

(a) *If $d^2l > r$, then $dL' - (E_1 + \cdots + E_r)$ is nef.*

(b) *If $d^2l \leq r$, then $d'L' - (E_1 + \cdots + E_{ld^2})$ is a nef $\mathbf{Q}$-divisor for all rational $d' > d$, and, for each integer $1 \leq j < d^2l$,*

$$d'L' - (E_1 + \cdots + E_j) - \frac{(d^2l - j)}{(r - j)}(E_{j+1} + \cdots + E_{\lfloor\lambda\rfloor} + (\lambda - \lfloor\lambda\rfloor)E_{\lceil\lambda\rceil})$$

is a nef $\mathbf{Q}$-divisor, where $\lambda = min\{r + (r - d^2l)(r - j)/(d^2l - j), n\}$ and $d' \geq d$ is any rational such that $d' > d$ if $\lambda = \lceil\lambda\rceil \leq n$.

Proof: We apply Lemma II.2. For (a), take $a_i = 1$ for $0 \leq i \leq r$ and $a_i = 0$ for $i > r$. For the first part of (b), take $a_0 = d'/d$ and $a_i = 1$ for $0 < i \leq d^2l$ and $a_i = 0$ for $i > d^2l$. For the rest, the idea is to choose a_i such that $a_i = 1$ for $0 \leq i \leq j$, with the a_i for $j < i \leq r$ being equal and as large as possible subject to $a_0d^2l \geq a_1 + \cdots + a_r$ (hence $a_i = (d^2l - j)/(r - j)$ for $j < i \leq r$), and finally for as many as possible of the remaining a_i also to equal $(d^2l - j)/(r - j)$, subject to $ra_0 \geq a_1 + \cdots + a_n$. Thus we take $a_i = (d^2l - j)/(r - j)$ for $r < i \leq \lfloor\lambda\rfloor$, $a_i = 0$ for $i > \lceil\lambda\rceil$ and, if $(\lambda - \lfloor\lambda\rfloor) > 0$, we take $a_{\lceil\lambda\rceil} = (\lambda - \lfloor\lambda\rfloor)(d^2l - j)/(r - j)$ (in which case $ra_0 \geq a_1 + \cdots + a_n$ will be an equality). The requirement on d' ensures positive self-intersection. $\diamondsuit$

III. Seshadri Constants of Very Ample Divisors

For a very ample divisor L on a surface X with $L^2 = l$ and any n distinct points p_i on X, it is easy to see that $\epsilon(L, p_1, \ldots, p_n) \leq \sqrt{l/n}$: just note that for any ε bigger than $\sqrt{l/n}$ we can find a rational $\delta < \sqrt{l/n}$ such that $F_\varepsilon \cdot F_\delta < 0$, where $F_t = L' - t(E_1 + \cdots + E_n)$. But $F_\delta^2 > 0$, so for appropriate integers N sufficiently large, $|NF_\delta|$ is nonempty, hence $\epsilon(L, p_1, \ldots, p_n) \leq \varepsilon$.

Lower bounds are more difficult. By applying Corollary II.3, we establish our main lower bound.

Theorem III.1: *Let $l = L^2$, where L is a very ample divisor on an algebraic surface X. Then $\sqrt{l/n} \geq \epsilon(L, n)$, and in addition, we have $\epsilon(L, n) \geq \varepsilon_{n,l}$ unless $l \leq n$ and nl is a square, in which case $\sqrt{l/n} = \varepsilon_{n,l}$ and $\epsilon(L, n) \geq \sqrt{l/n} - \varepsilon$ for every positive rational ε.*

Proof: We noted $\sqrt{l/n} \geq \epsilon(L, n)$ above. If $l > n$, then $\varepsilon_{n,l} = 1$ by Proposition I.2, but $L' - E_1 - \cdots - E_n$ is nef by Corollary II.3(b) (take $r = n$ and $d = 1$), so $\epsilon(L, n) \geq 1$. If $l < n$ but nl is not a square, then $\epsilon(L, n) \geq \varepsilon_{n,l}$ follows from Corollary II.3, parts (a) and (b). Finally, if $l \leq n$ and nl is a square, then $\sqrt{l/n} = \varepsilon_{n,l}$ follows from Proposition I.2, and $\epsilon(L, n) \geq \sqrt{l/n} - \varepsilon$ holds for every positive rational ε by Corollary II.3(c). $\diamondsuit$

Although one needs to check only finitely many values of r and d to compute $\varepsilon_{n,l}$, it is nonetheless useful to have more explicit lower bounds. For that purpose, given positive integers n and l, let $d^* = \lceil \sqrt{n/l} \rceil$, $d_* = \lfloor \sqrt{n/l} \rfloor$, $r^* = \lceil d_* \sqrt{nl} \rceil$, and $r_* = \lfloor d_* \sqrt{nl} \rfloor$.

Corollary III.2: *Let l and n be positive integers. Then $\varepsilon_{n,l} \geq 1/d^*$, and, if $l \leq n$, then also $\varepsilon_{n,l} \geq \max(r_*/(nd_*), d_* l/r^*)$.*

Proof: For the first inequality, use $r = n$ and $d = d^*$, and check that then $1 \leq r \leq n$, $1 \leq d$, and $r/d \leq \sqrt{nl}$, so in this case $r/(nd) \in S_1$. For $r_*/(nd_*)$ in the second inequality, use $r = r_*$ and $d = d_*$, and again check that $1 \leq r \leq n$, $1 \leq d$ (because $l \leq n$), and $r/d \leq \sqrt{nl}$, so $r/(nd) \in S_1$. For $d_* l/r^*$, use $r = r^*$ and $d = d_*$, and check that $1 \leq r \leq n$, $1 \leq d$, and $r/d \geq \sqrt{nl}$, so $dl/r \in S_2$. $\diamondsuit$

Remark III.3: The values of r and d obtained using d^*, d_*, r^*, and r_* are not always optimal. For example, if $n = 33$ and $l = 1$, then $\varepsilon_{n,l} = 4/23 \in S_2(n,l)$ comes from $r = 23$ and $d = 4$, but $1/d^* = 1/6$, $r_*/(nd_*) = 28/(33 \cdot 5)$, $d_*/r^* = 5/29$ are all less than $4/23$.

Remark III.4: If L is a line in $X = \mathbf{P}^2$, then $l = 1$. If we denote $\lfloor \sqrt{n} \rfloor$ by s and $\lfloor (n - s^2)/2 \rfloor$ by t, then either $n = s^2 + 2t$ or $n = s^2 + 2t + 1$, where $0 \leq t \leq s$ (with $t < s$ in the latter case). With respect to s and t in the case that n is not a perfect square, it is not hard to check that $r^* = s^2 + t$, $r_* = s^2 + t - 1$, $d^* = s + 1$ and $d_* = s$ if $n = s^2 + 2t$, while $r^* = s^2 + t + 1$, $r_* = s^2 + t$, $d^* = s + 1$ and $d_* = s$ if $n = s^2 + 2t + 1$. If $n = s^2$, then $r^* = r_* = s^2$ and $d^* = d_* = s$. (For a more symmetrical treatment, under some restrictions, of cases (a) and (b) of the following corollary, see Example IV.7.)

Corollary III.5: *Let $1 \leq s$ and $0 \leq t \leq s$ be integers.*
- *(a) If $n = s^2 + 2t$, then $\varepsilon_n \geq s/(s^2 + t)$.*
- *(b) If $n = s^2 + 2t + 1$ and $t < s$, then $\varepsilon_n \geq (s^2 + t)/(s(s^2 + 2t + 1))$*
- *(c) If $s > 1$ and $n = s^2 + 2t + 1$ and $0 < t < (\sqrt{2} - 1)(s - 1)$, then $\varepsilon_n \geq (s(s-1) + t)/((s-1)(s^2 + 2t + 1)) > (s^2 + t)/(s(s^2 + 2t + 1))$.*
- *(d) If $s > 1$ and $n = s^2 + 2t + 1$ and $(\sqrt{2} - 1)(s - 1) < t < \sqrt{1.25s^2 - s} - s/2$, then $\varepsilon_n \geq (s-1)/(s(s-1) + t) > (s^2 + t)/(s(s^2 + 2t + 1))$.*

Proof: For (a) and (b), apply Corollary III.2, using the expressions for r^*, r_*, d^* and d_* in Remark III.4. For (c) and (d), apply Corollary II.3(b) and (a), resp., with $r = s(s-1) + t$ and $d = s - 1$.$\diamondsuit$

Remark III.6: For L a line in $X = \mathbf{P}^2$ over $\mathbf{C}$, [ST] proves for $n \geq 10$ that $\epsilon(L, n) \geq 1/\sqrt{n+1}$. It is easy to check that Corollary III.5 gives a better result in all cases except when $n \pm 1$ is a square. For improvements in these cases, see Proposition I.3.

Remark III.7: The lower bounds given in Corollary III.5(a,b) actually equal both $\epsilon(L, n)$ and ε_n if $1 \leq n \leq 6$. For $n = 7$, $\epsilon(L, 7) = 3/8$ (since $F = 8L' - 3E_1 - \cdots - 3E_7$ is known to be nef while $E = 3L' - 2E_1 - E_2 - \cdots - E_7$ is effective with $F \cdot E = 0$), and $\epsilon(L, 8) = 6/17$ for $n = 8$ (since $F = 17L' - 6E_1 - \cdots - 6E_8$ is nef while $E = 6L' - 3E_1 - 2E_2 - \cdots - 2E_8$ is effective with $F \cdot E = 0$), whereas in fact $\varepsilon_7 = 5/14$ and $\varepsilon_8 = 1/3$.

Proof of Proposition I.2: Part (c) is easy to check using Corollary III.5(a,b). For part

(a), note that we have $\varepsilon_{n,l} \geq 1/d^*$ by Corollary III.2, but $d^* = 1$ for $l \geq n$. On the other hand, by definition either $\varepsilon_{n,l} = r/(nd)$ for some positive r and d with $r \leq n$ (in which case clearly $\varepsilon_{n,l} \leq 1$), or $\varepsilon_{n,l} = dl/r$ for some positive r and d with $r \leq n$ and $r^2 \geq d^2nl$ (hence $dl/r = d^2nl/(rdn) \leq r^2/(rdn) \leq 1$).

Consider part (b). Given r and d with $\delta = r^2 - d^2nl$, it is easy to check that $dl/r = \sqrt{l/n}\sqrt{1-\delta/r^2}$ if $0 \leq \delta$, while $r/(nd) = \sqrt{l/n}\sqrt{1+\delta/(d^2nl)}$ if $\delta \leq 0$. The inequalities in (b)(ii) now follow by definition of $\varepsilon_{n,l}$. Moreover, this makes it clear that $\varepsilon_{n,l} \leq \sqrt{l/n}$, so if $nl = q^2$ for some q, we can take $r = q$ and $d = 1$ to see $\varepsilon_{n,l} \geq r/(nd) = \sqrt{l/n}$. This proves part (b)(i). To prove the statement about equality in (b)(ii), first assume $r^2 - nld^2 = 1$ with $r \leq n$. It suffices to show $\varepsilon_{n,l} = dl/r$.

For any positive integer $t \leq \sqrt{n/l}$, denote $\lceil t\sqrt{nl}\rceil$ by r_t; e.g., we have $r = r_d$. Since $r^2 - nld^2 = 1$, we know that nl is not a perfect square, so $\lfloor t\sqrt{nl}\rfloor = r_t - 1$. Now, $\varepsilon_{n,l}$ is just the maximum in $\{(r_t-1)/(tn) | 1 \leq t \leq \sqrt{n/l}\} \cup \{1/t | t = \lceil\sqrt{n/l}\rceil\} \cup \{tl/r_t | 1 \leq t \leq \sqrt{n/l}\}$. We will show that $r_t = \lceil rt/d\rceil$. Assuming this, it follows that $dl/r = tl/(rt/d) \geq tl/r_t$, so dl/r is the maximum of $\{tl/r_t | 1 \leq t \leq \sqrt{n/l}\}$. We must also show dl/r is as large as every element of $\{(r_t-1)/(tn) | 1 \leq t \leq \sqrt{n/l}\} \cup \{1/\lceil\sqrt{n/l}\rceil\}$. But from $r^2 - nld^2 = 1$ we derive $r^2/(d^2l^2) = n/l + 1/(d^2l^2)$. If $dl/r < 1/\lceil\sqrt{n/l}\rceil$, then $\lceil\sqrt{n/l}\rceil < r/(dl)$, and there must be an integer k with $\sqrt{n/l} \leq k < r/(dl)$, hence $r^2/(d^2l^2) - 1/(d^2l^2) = n/l \leq k^2 < r^2/(dl)^2$, and so $r^2 - 1 \leq k^2d^2l^2 < r^2$, which is absurd. As for $\{(r_t-1)/(tn) | 1 \leq t \leq \sqrt{n/l}\}$, we have $rt = r_td - \rho$ where $0 \leq \rho < d$. By solving for r_t and substituting, we see $(r_t-1)/(nt) \leq dl/r$ if and only if $t = t(r^2 - nld^2) \leq (d-\rho)r$. But $(d-\rho)r \geq r > d\sqrt{nl} \geq \sqrt{nl} \geq \sqrt{n/l} \geq t$, as required.

We are left with checking $r_t = \lceil rt/d\rceil$. Since $r = \lceil d\sqrt{nl}\rceil$, we see $rt \geq dt\sqrt{nl}$, so $r_t = \lceil rt/d\rceil$ follows if we show there is no integer k with $t\sqrt{nl} < k < rt/d$ (equivalently, that there is no k with $t^2d^2nl < k^2d^2 < r^2t^2$), but such a k would imply $t^2d^2nl < (rt-1)^2$. Thus it suffices to show that $rt - dt\sqrt{nl} \leq 1$ for $1 \leq t \leq \sqrt{n/l}$. But $t(r - d\sqrt{nl}) = t(r^2 - nld^2)/(r + d\sqrt{nl}) < t/(2d\sqrt{nl}) \leq \sqrt{n/l}/(2d\sqrt{nl}) = 1/(2dl) \leq 1$.

Suppose now that $r^2 - nld^2 = -1$, in which case we must show $\varepsilon_{n,l} = r/(nd)$. This time $r_t = \lfloor t\sqrt{nl}\rfloor$, $\lceil t\sqrt{nl}\rceil = r_t + 1$ and we will show $r_t = \lfloor rt/d\rfloor$. It follows that $r/(nd) = rt/(ndt) \geq r_t/(nt)$. We also have $r/(nd) \geq 1/\lceil\sqrt{n/l}\rceil$: if not then $\lceil\sqrt{n/l}\rceil < nd/r$, but $n/l = (dn/r)^2 - n/(r^2l)$, so there is an integer k with $(dn/r)^2 - n/(r^2l) = n/l \leq k^2 < (dn/r)^2$, hence $(dn)^2 - n/l \leq k^2r^2 < (dn)^2$, but this is not sufficient distance between squares unless $d = n = l = 1$, which contradicts $r^2 - d^2nl = -1$. Now compare $r/(nd)$ with $tl/(r_t+1)$. We have $rt = r_td + \rho$ where $0 \leq \rho < d$, and as before $tl/(r_t+1) \leq r/(nd)$ if and only if $t \leq (d-\rho)r$. But $(d-\rho)r \geq r > d\sqrt{nl} - 1 = dl\sqrt{n/l} - 1$, and $dl\sqrt{n/l} - 1 \geq t$ unless $t = \lfloor\sqrt{n/l}\rfloor$ and $dl = 1$, but in that case it is easy to check that $t = r - 1$.

We are left with checking $r_t = \lfloor rt/d\rfloor$. Since $r = \lfloor d\sqrt{nl}\rfloor$, we see $rt \leq dt\sqrt{nl}$, so $r_t = \lfloor rt/d\rfloor$ follows if $dt\sqrt{nl} - rt \leq 1$ for $1 \leq t \leq \sqrt{n/l}$. But $t(d\sqrt{nl} - r) = t(nld^2 - r^2)/(d\sqrt{nl} + r) < t/(d\sqrt{nl}) \leq \sqrt{n/l}/(dl\sqrt{n/l}) \leq 1/(dl) \leq 1$.

Finally, consider part (b)(iii). There exist r and d such that either $\varepsilon_{n,l} = dl/r$ with $0 \leq \delta = r^2 - d^2nl$ or $\varepsilon_{n,l} = r/(nd)$ with $\delta \leq 0$. If $0 \leq \delta$, it's enough to check that $dl/r > \sqrt{l/n}\sqrt{1-1/n}$, but as above $dl/r = \sqrt{l/n}\sqrt{1-\delta/r^2}$ so it suffices to check that $\delta/r^2 < 1/n$; i.e., that $\delta < r^2/n$. If $\delta \leq 0$, the argument is the same except $\varepsilon_{n,l} = r/(nd) =$

$\sqrt{l/n}\sqrt{1+\delta/(d^2nl)}$.

So, to bound the number of l for which $(*)$ in (b)(iii) holds, we check whether $-d^2l < \delta < r^2/n$ holds when $d = 1$ and r is either $r = \lfloor\sqrt{nl}\rfloor$ or $r = \lceil\sqrt{nl}\rceil$. But $-d^2l < \delta < r^2/n$ holds if either $\lceil\sqrt{nl}\rceil < \sqrt{nl}/\sqrt{1-1/n}$ or $\lfloor\sqrt{nl}\rfloor > \sqrt{nl}\sqrt{1-1/n}$, which is equivalent to having the interval $I_l = (\sqrt{nl}\sqrt{1-1/n}, \sqrt{nl}/\sqrt{1-1/n})$ contain an integer. It is not too hard to check that the union $I_{\lceil(n-1)/2\rceil} \cup \cdots \cup I_{n-1}$ contains the interval $(\sqrt{n(n-1)/2}, n-1)$, and thus the number of values of l between $(n-1)/2$ and $n-1$ for which $\varepsilon_{n,l} > \sqrt{l/n}(\sqrt{1-1/n})$ holds is always at least $(n-1) - \sqrt{n(n-1)/2} - 1$. This is at least half of the number of l in the range $(n-1)/2 \leq l < n$, as long as $n \geq 45$. An explicit check for $3 \leq n \leq 44$ shows that $(*)$ still holds for at least half of the number of l in the range $(n-1)/2 \leq l < n$.

For $(**)$, we apply Dirichlet's theorem from elementary number theory, which says there are integers $0 < r < n+1$ and $d \geq 1$ such that $|r/\sqrt{nl} - d| \leq 1/(n+1)$. Given such an r and d, we have $|r - d\sqrt{nl}| \leq \sqrt{nl}/(n+1)$, hence $|\delta| = |r^2 - nld^2| \leq (r + d\sqrt{nl})\sqrt{nl}/(n+1)$. Thus, if $\delta < 0$, we have $|\delta|/(nld^2) < \sqrt{nl}(r + d\sqrt{nl})/(n^2ld^2) < \sqrt{nl}(2d\sqrt{nl})/(n^2ld^2) = 2/(nd)$, and if $\delta > 0$, we have $\delta/r^2 < \sqrt{nl}(r + d\sqrt{nl})/(nr^2) < \sqrt{nl}(2r)/(nr^2) < (r/d)(2r)/(nr^2) = 2/(nd)$.

For some $r = r'$ and $d = d'$ and $\delta = r'^2 - nld'^2$, we know that $\varepsilon_{n,l} = \sqrt{l/n}\sqrt{1-x}$, where $x = |\delta|/(nld'^2)$ if $\delta < 0$ and $x = \delta/r'^2$ if $\delta > 0$, and thus either way $x < 2/(nd')$. It follows that if, for the given n, $|r/\sqrt{nl} - d| \leq 1/(n+1)$ holds for no r and d with $0 < r < n+1$ and $1 \leq d < 2a$, then $x < 2/(nd') \leq 1/(an)$, and hence $(**)$ holds for this n. So to count those n in the range $s^2l \leq n < (s+1)^2l$ for which $(**)$ holds, it is enough to count how often $|r/\sqrt{nl} - d| \leq 1/(n+1)$ holds for $1 \leq d < 2a$ and $0 < r < n+1$.

But for any given d, $|r/\sqrt{nl} - d| \leq 1/(n+1)$ holds for some r only if the interval $(d\sqrt{nl} - \sqrt{l/n}, d\sqrt{nl} + \sqrt{l/n})$ contains an integer, i.e., only if $[d^2nl - 2dl + 1, d^2nl + 2dl]$ contains a square. Now, the interval $[d^2s^2l^2, d^2(s+1)^2l^2]$ contains $dl + 1$ squares, and we are interested in counting for how many $n \in [s^2l, (s+1)^2l)$ does the interval $[d^2nl - 2dl + 1, d^2nl + 2dl]$ contain one of these squares. Since $s > 2$, no interval $[d^2nl - 2dl + 1, d^2nl + 2dl]$ can contain two squares, and for $d > 3$, the intervals are disjoint and $d^2(s+1)^2l^2$ is in no interval, so at most dl of the intervals contain squares. For $2 \leq d \leq 3$, consecutive intervals overlap but no point lies in three intervals (and $d^2(s+1)^2l^2$ is in no interval if $d = 3$ and only in the last interval if $d = 2$), so there are at most $2dl$ intervals that contain squares when $d = 3$ and at most $2dl + 1$ when $d = 2$. Similarly, for $d = 1$ at most four intervals overlap at a single point and $d^2(s+1)^2l^2$ is in two intervals, so there are at most $4dl + 2$ intervals that contain squares. (These are of course typically overestimates since some squares may lie in no intervals.) Summing over $1 \leq d < 2a$, we find that of the n in the range $s^2l \leq n < (s+1)^2l$ there are at most $(4l + 2 + 4l + 1 + 6l) + (4l + 5l + \cdots + (2a-1)l)$ (i.e., $(2a^2 - a + 8)l + 3$ if $a > 2$, $14l + 3$ if $a = 2$, and $4l + 2$ if $a = 1$) values of n whose corresponding interval contains a square. $\diamondsuit$

Remark III.8: Our estimate that $(*)$ in Proposition I.2(b) holds for at least half of $(n-1)/2 \leq l < n$ understates how often $(*)$ holds. One reason for this is that the intervals I_l in the proof overlap, and thus the same integer can lie in more than one interval, but our estimate counts only some of those integers, and at most once each. Also, our estimate is based on a check only for $d = 1$. We can partially account for cases with $d > 1$ by

again applying Dirichlet's theorem, as in the proof of Proposition I.2(b)(iii). As we saw there, $-2dl < \delta < 2r^2/(nd)$ holds for any r and d such that $|r/\sqrt{nl} - d| \leq 1/(n+1)$ with $0 < r < n+1$ and $d \geq 1$. Therefore, $-d^2l < \delta < r^2/n$ also holds if in addition $d > 1$. Thus, as long as $|r/\sqrt{nl} - 1| > 1/(n+1)$ holds for $r = \lfloor\sqrt{nl}\rfloor$ and $r = \lceil\sqrt{nl}\rceil$ (which we can rewrite as $(n+2)\sqrt{nl}/(n+1) - 1 < \lfloor\sqrt{nl}\rfloor < n\sqrt{nl}/(n+1)$), we see that the solution to $|r/\sqrt{nl} - d| \leq 1/(n+1)$ guaranteed by Dirichlet's theorem must have $d > 1$ and hence $(*)$ holds. I.e., if the interval $J_l = ((n+2)\sqrt{nl}/(n+1) - 1, n\sqrt{nl}/(n+1))$ contains an integer, then $\varepsilon_{n,l} > \sqrt{l/n}\sqrt{1 - 1/n}$.

One can check that the intervals J_l are nonempty as long as l is less than about $n/4$, and that these intervals and the I_l are all disjoint as long as l is less than about $n/2$, and that the union of the I_l for l more than about $n/2$ is about $(n/\sqrt{2}, n)$. Thus the union of all of the intervals I_l and J_l has measure about $0.61n$, so it is reasonable (but not guaranteed) to expect that at least 61% of the values of l from 1 to n should give $\varepsilon_{n,l} > \sqrt{l/n}\sqrt{1 - 1/n}$. To take into account overlaps among the I_l for $l > n/2$, we might instead want to consider the sum of the lengths of the intervals. This is about $3n/4$, and so it is reasonable to expect that typically at least 75% of the values of l from 1 to n result in $\varepsilon_{n,l} > \sqrt{l/n}\sqrt{1 - 1/n}$. Explicit computations for various n show, in fact, that percentages around 80% are common. For n from 15 to 200, the smallest percentage (63%) occurs for $n = 19$ and the largest (87.6%) for $n = 97$. For some larger n, we have 85% for $n = 313$, 75% for $n = 314$, 78% for $n = 3079$ and 80.8% for $n = 3080$.

We now prove Corollary I.4 and Corollary I.5.

Proof of Corollary I.4: This follows from Corollary II.3(a), if we check that $r^2 > b^2(a^2n)l$ and $r \leq a^2n$. But $r^2 = 1 + d^2nl > d^2nl = (ab)^2nl$, and $r^2 - 1 = d^2nl = a^2nb^2l = a^2n^2b^2l/n < a^4n^2$, hence $r^2 \leq a^4n^2$, as required. $\diamondsuit$

Proof of Corollary I.5: For the first part it is clearly enough to consider the case that $r = \lceil d\sqrt{nl}\rceil$, and apply Corollary II.3(a,c). Similarly, for any rational $r > \lceil d\sqrt{nl}\rceil$, H_t is nef for any rational $r > t > \lceil d\sqrt{nl}\rceil$ by Corollary II.3(a,c), hence $H_r = (r-t)L' + H_t$ is ample (since L' meets every curve positively except for $E_1, \ldots, E_n$, which H_t meets positively). $\diamondsuit$

IV. Refinements

The bound $\epsilon(L, n) \geq \varepsilon_{n,l}$ given in Theorem I.1 is limited by the requirement in the definition of $\varepsilon_{n,l}$ that $r \leq n$. To get stronger results we need to relax this requirement. Our definition of $\varepsilon_{n,l}$ is based on Corollary II.3, which in turn is based on constructing nef divisors by blowing up a smooth point of an irreducible curve linearly equivalent to a multiple of a very ample divisor. Considering singular points allows us, in effect, to use values of r that can be bigger than n.

For example, say m is a positive integer, p'_1 is a smooth point of an algebraic surface X, and $X_{p'_1}$ is the blowing up of X at p'_1, with E being the corresponding exceptional divisor. If L is very ample on X, then $tL' - mE = (t-m)L' + m(L' - E)$ is very ample on $X_{p'_1}$ for any $t > m > 0$, where L' is the pullback of L to $X_{p'_1}$. Thus $|tL' - mE|$ contains

an element C_1 which is reduced and irreducible and is smooth and transverse to E at some point $p'_2 \in E$. Given the morphism $\pi' : Y' \to X$ corresponding to the proximity sequence $p'_1, \ldots, p'_n$ with p'_1 and p'_2 as above and each p'_i, $i \leq r$, being infinitely near points on proper transforms of C_1, we find that $[d(\pi'^*L) - mE'_1 - E'_2 - \cdots - E'_n]$ is the class of an irreducible divisor (in fact, the proper transform of C_1) on Y'. Define the function $f(d) = \max(1, d-1)$; applying Lemma II.1 in the same manner as in Corollary II.3 we obtain:

Corollary IV.1: *Given X, Y, l, n and L' as in Lemma II.2, let $1 \leq d$, $1 \leq m \leq f(d)$ and $1 \leq r \leq n$ be integers. Then we have the following cases.*

(a) *If $(r+m-1)^2 > nd^2l$, then $(r+m-1)L' - dl(E_1 + \cdots + E_n)$ is nef.*

(b) *If $(r+m-1)^2 < nd^2l$, then $ndL' - (r+m-1)(E_1 + \cdots + E_n)$ is nef.*

(c) *If $(r+m-1)^2 = nd^2l$, then $tL' - (E_1 + \cdots + E_n)$ is a nef $\mathbf{Q}$-divisor for all rationals $t > \sqrt{n/l}$.*

If we now define the sets

$$S'_1(n,l) = \left\{ \frac{r+m-1}{nd} \;\middle|\; 1 \leq r \leq n,\ 1 \leq d,\ 1 \leq m \leq f(d),\ \frac{r+m-1}{d} \leq \sqrt{nl} \right\},$$

$$S'_2(n,l) = \left\{ \frac{dl}{r+m-1} \;\middle|\; 1 \leq r \leq n,\ 1 \leq d,\ 1 \leq m \leq f(d),\ \frac{r+m-1}{d} \geq \sqrt{nl} \right\},$$

and $S'(n,l) = S'_1(n,l) \cup S'_2(n,l)$, we can take $\varepsilon'_{n,l} = \max(S'(n,l))$. Note that since we can rewrite S'_1 and S'_2 as $S'_1(n,l) = \{r/(nd) | 1 \leq r \leq n + f(d) - 1, 1 \leq d, r/d \leq \sqrt{nl}\}$ and $S'_2(n,l) = \{dl/r | 1 \leq r \leq n + f(d) - 1, 1 \leq d, r/d \geq \sqrt{nl}\}$, this effectively allows us to use r bigger than n. With essentially the same proof as for Theorem III.1, we now have:

Theorem IV.2: *Let $l = L^2$, where L is a very ample divisor on an algebraic surface X. Then $\sqrt{l/n} \geq \epsilon(L,n)$, and in addition, we have $\epsilon(L,n) \geq \varepsilon'_{n,l}$ unless $l \leq n$ and nl is a square, in which case $\sqrt{l/n} = \varepsilon'_{n,l}$ and $\epsilon(L,n) \geq \sqrt{l/n} - \varepsilon$ for every positive rational ε.*

Example IV.3: This actually is only a minor improvement, but it is an improvement. For example, if $n+2$ is a square, then we can write $n = s^2 + 2s - 1$ for some $s \leq n$. If $s \geq 2$, then apply Corollary IV.1(a) with $r = n$, $m = 2$ and $d = s+1$ to see that $(r+m-1)L' - d(E_1 + \cdots + E_n)$ is nef, and hence $\epsilon(L,n) \geq \varepsilon'_{n,1} \geq d/(r+m-1) = (s+1)/(s^2+2s) = \sqrt{1/n}\sqrt{1 - 1/(n+1)^2}$. This is better than what we got before (cf. Corollary III.5), and in fact is precisely the bound obtained in [Bi] for $n = a^2i^2 - 2i$ for $i = 1$.

We can get a further effective increase in r by considering additional, infinitely near singularities. For example, we have:

Corollary IV.4: *Say L is a line in $X = \mathbf{P}^2$, and consider positive integers $d \geq 4$, $n \geq 5$, $1 \leq r' \leq n+d-1$. Then, for a blowing up of n general points of $\mathbf{P}^2$, $r'L' - d(E_1 + \cdots + E_n)$ is nef if $r'^2 > nd^2$, and $ndL' - r'(E_1 + \cdots + E_n)$ is nef if $r'^2 < nd^2$.*

Proof: If $r' \leq n+d-2$, then we can take $r \leq n$ and $m \leq d-1$ but still have $r+m-1 = r'$, so the result follows by Corollary IV.1. Thus we may assume that $r' = n+d-1$. The idea is to

choose a proximity sequence $p'_1, \ldots, p'_n$ such that $[dL'-(d-2)E'_1-2E'_2-2E'_3-E'_4-\cdots-E'_n]$ is the class of an irreducible effective divisor. Given this the result follows from Lemma II.1.

To justify our claim about $[dL' - (d-2)E'_1 - 2E'_2 - 2E'_3 - E'_4 - \cdots - E'_n]$, we first pick $p'_1, \ldots, p'_4$, such that p'_2 is on the exceptional divisor of the blow up of p'_1, p'_3 is a general point on the exceptional divisor of the blow up of p'_2 (hence not on the proper transforms of the line through p'_1 and p'_2 nor of the exceptional divisor of p'_1), and p'_4 is a general point on the exceptional divisor of the blow up of p'_3. The claim is now that $[dL' - (d-2)E'_1 - 2E'_2 - 2E'_3 - E'_4]$ is the class of a reduced irreducible divisor C_4 meeting E_4 transversely. To see this, note that this class corresponds under a quadratic Cremona transformation centered at p'_1, p'_2, p'_3 to the class $[(d-2)L'' - (d-4)E''_1 - E''_4]$, where the E''_i are obtained by blowing up four points with the first three as before and the fourth point being a general point on the line through p'_1 and p'_2, but not infinitely near to any of the first three. But clearly for any $d \geq 4$ there is a reduced irreducible curve of degree $d-2$ with a $(d-4)$-multiple point passing simply through some other general point. To finish picking our proximity sequence, let $p'_5, \ldots, p'_n$ be the points of the proper transforms of C_4 infinitely near to p'_4.$\diamondsuit$

Example IV.5: As an application of the previous result, let L be a line in $\mathbf{P}^2$. For $8 \leq n = s^2 + 2s$ (thus, $n+1$ is a square), we have $\epsilon(L, n) \geq (s^2 + 3s + 1)/(s(s+2)^2) = \sqrt{1/n}\sqrt{1 - (n-1)/(n(\sqrt{n+1}+1)^2)}$, and if $10 \leq n = s^2 + 1$ (i.e., $n-1$ is a square), we have $\epsilon(L, n) \geq (s+1)/(s^2+s+1)) = \sqrt{1/n}\sqrt{1-(n-1)/(n+\sqrt{n-1})^2}$. To see this, apply Corollary IV.4: for $n = s^2 + 2s$, take $r = n$, $m = d-2 = s$ and $r' = r + m + 1$ and note $r'^2 < nd^2$, and for $n = s^2 + 1$, take $r = n$, $m = d - 2 = s - 1$ and $r' = r + m + 1$ and note $r'^2 > nd^2$.

We can also obtain additional improvements in our bounds in special cases, based on the following result.

Lemma IV.6: *Let $d = abc$, where a, b, c are positive integers with $c < a$ such that c and a are relatively prime and the characteristic does not divide c. If $r' \geq a^2b^2c$ and $n \geq a^2b^2 + (r' - a^2b^2c)$ are integers, then, for a blowing up of n general points of $\mathbf{P}^2$ with $L \subset \mathbf{P}^2$ a line, $r'L' - d(E_1 + \cdots + E_n)$ is nef if $r'^2 > nd^2$, and $ndL' - r'(E_1 + \cdots + E_n)$ is nef if $r'^2 < nd^2$.*

Proof: Let $r = a^2b^2 + (r' - a^2b^2c)$. The idea is to show there is a proximity sequence $p'_1, \ldots, p'_n$ such that $dL' - c(E'_1 + \cdots + E'_{a^2b^2}) - (E'_{a^2b^2+1} + \cdots + E'_r)$ is linearly equivalent to an irreducible divisor, then apply Lemma II.1: if $r'^2 > nd^2$, take $a_0 = r'/d$ and $a_1 = \cdots = a_n = d$, while if $r'^2 < nd^2$, take $a_0 = n$ and $a_1 = \cdots = a_n = r'$.

Now we construct our irreducible divisor $dL' - c(E'_1 + \cdots + E'_{a^2b^2}) - (E'_{a^2b^2+1} + \cdots + E'_r)$. We will be very explicit. Choose homogeneous coordinates x, y, z on $\mathbf{P}^2$, let $G = xz^{cb-1} - y^{cb}$, let $F = x^{ab} + z^{(a-c)b}G$. Note that G and F meet only at $p'_1 = [0:0:1]$ (with order of contact therefore ab^2c), and both F and G are smooth at p'_1. It follows that F and G are reduced and irreducible, and p'_1 is a base point of the pencil $\langle F^c, G^a \rangle$. This pencil gives a rational map to $\mathbf{P}^1$. By successively blowing up points, we can remove the indeterminacies

of this map. It turns out that the points one must blow up to do so give a proximity sequence $p'_1, \ldots, p'_{a^2b^2}$. On the blow up the rational map is a morphism, and the class of the fiber of the morphism corresponding to F^c is just $[cC] = [abcL' - cE'_1 - \cdots - cE'_{a^2b^2}]$, where C is the proper transform of the curve defined by F. The fiber corresponding to G^a is $aD + (a-c)N_1 + 2(a-c)N_2 + \cdots + ab^2c(a-c)N_{ab^2c} + (ab^2c(a-c)-c)N_{ab^2c+1} + (ab^2c(a-c)-2c)N_{ab^2c+2} + \cdots + cN_{a^2b^2-1}$ (call this divisor A for short) where D is the proper transform of the curve defined by G, and N_i is the effective divisor whose class is $[E'_i - E'_{i+1}]$. Thus A and cC move in a base point free pencil defining a morphism to $\mathbf{P}^1$. The divisor $E'_{a^2b^2}$ is a multisection of this morphism, since it meets each fiber c times. By Bertini's Theorem (see Lemma II.6 of [H1]), the general member is either reduced and irreducible or every member is a sum of c elements of $|C|$. But the latter would imply that A is a sum of c members of $|C|$; A is connected so A would have to be c times a single element of $|C|$, which is impossible since D is a component of A of multiplicity a, and a and c are relatively prime. Moreover, the trace of the fibers of the morphism on $E'_{a^2b^2}$ is a linear system spanned by two points of multiplicity c (since A and cC both meet $E'_{a^2b^2}$ at single points with multiplicity c). Since the characteristic does not divide c, some general fiber H is reduced and irreducible and meets $E'_{a^2b^2}$ transversely. Now take $p'_{a^2b^2+1}$ to be one of these transverse points of intersection; this uniquely determines the rest of the proximity sequence through p'_r, with respect to which $[dL' - c(E'_1 + \cdots + E'_{a^2b^2}) - (E'_{a^2b^2+1} + \cdots + E'_r)]$ is the class of the proper transform of H, which is irreducible. The rest of the proximity sequence can be chosen arbitrarily, as long as we don't blow up any more points of H and keep $E'_i - E'_{i+1}$ irreducible.$\diamondsuit$

Example IV.7: Let $n = s^2 + j$, where s and j are positive integers. If we assume s is not a power of 2 and that the characteristic is not 2, then we may take $c = 2$, a to be any odd prime factor of s, $b = s/a$, $d = abc = 2s$ and $r' = ca^2b^2 + i$, where i is an integer $0 \leq i \leq j$. We find $\delta = r'^2 - nd^2 = a^2b^2c(2i - cj) + i^2$. This satisfies the hypotheses of Lemma IV.6, so either $r'L' - d(E_1 + \cdots + E_n)$ or $ndL' - r'(E_1 + \cdots + E_n)$ is nef, depending on the sign of δ. If we take $i = j$, it follows that $r'L' - d(E_1 + \cdots + E_n)$ is nef and hence that $\epsilon(L, n) \geq d/r' = \sqrt{1/n}\sqrt{1 - i^2/(2s^2+i)^2} = \sqrt{1/n}\sqrt{1 - i^2/(2n-i)^2}$. When $i = 1$ (and hence $n - 1$ is a square), this is the bound given in [Bi] over $\mathbf{C}$ (but with no restriction on s), but this remains a very good bound as long as i is not too big. Similarly, if we take $i = j - 1 \leq 2s - 1$, then $ndL' - r'(E_1 + \cdots + E_n)$ is nef and $\epsilon(L, n) \geq r'/(nd) = \sqrt{1/n}\sqrt{1 - (4s^2 - i^2)/(4ns^2)}$. This bound is especially good when i is near $2s$. For example, if $i = 2s - 1$ (and hence $n + 1$ is a square) we have $\epsilon(L, n) \geq \sqrt{1/n}\sqrt{1 - (4s-1)/(4ns^2)} > \sqrt{1/n}\sqrt{1 - (4s)/(4ns^2)} = \sqrt{1/n}\sqrt{1 - 1/(n(\sqrt{n+1} - 1))}$.

Proof of Proposition I.3: The claims of Proposition I.3 are proved by Example IV.3, Example IV.5 and Example IV.7. $\diamondsuit$

V. Applications

Our results in Section II have numerous applications to questions of effectivity, regularity, base point freeness, ampleness and very ampleness for linear systems on $\mathbf{P}^2$.

In this section we will always let L be a line in $X = \mathbf{P}^2$ and take $\pi : Y \to X$ to be

the blow up of X at n general points $p_1, \dots, p_n$. Let E_i, $1 \leq \dots \leq n$, be the corresponding exceptional divisors and let $L' = \pi^* L$. Given $m > 0$, let $F_t = tL' - m(E_1 + \cdots + E_n)$. We can ask:

(a) What is the least t such that $|F_t|$ is nonempty?
(b) What is the least t such that F_t is ample?
(c) What is the least t such that F_t is regular (i.e., $h^1(Y, \mathcal{O}_Y(F_t)) = 0$)?
(d) What is the least t such that $|F_t|$ is base point free?
(e) What is the least t such that F_t is very ample?

For the rest of this section, F_t will be as above.

V.1. Effectivity

Here we consider question (a); i.e., what is the least t such that F_t is (linearly equivalent to) an effective divisor?

Corollary V.1.1: *If F_t is effective, then $t \geq mn\varepsilon_n$.*

Proof: By semicontinuity, F_t remains effective under specialization of the points, but $N = L' - \varepsilon_n(E_1 + \cdots + E_n)$ is nef for some choice of the points, hence $F_t \cdot N \geq 0$. ◊

In terms of simplicity, computability and being characteristic free, in addition to its being a very good bound in an absolute sense, this bound seems to be the best, overall, now known, at least for uniform multiplicities. Of course, for $D_t = tL' - m_1E_1 - \cdots - m_nE_n$ to be effective, it is true that $t \geq (m_1 + \cdots + m_n)\varepsilon_n$, but better bounds can sometimes be found. For example, if $D_t = tL' - 2m(E_1 + \cdots + E_7) - m(E_8 + \cdots + E_{15})$ is effective, then Corollary II.4(b) (with $d = 3$, $j = 7$, $r = 11$ and $n = 15$) gives $t \geq 6m$, whereas the fact that N in the proof of Corollary V.1.1 is nef gives only $t \geq 22\varepsilon_n m = 5.5m$. In some cases of nonuniform multiplicities, reduction by Cremona transformations can even give sharp bounds (see [H3]).

Even in the uniform case, there are special cases where better bounds are known, such as the calculation $\epsilon(L, p_1, \dots, p_{19}) \geq 39/170$ in [Bi] or the examples in Section IV. However, methods which bound effectivity by testing against nef or ample divisors can at best say $t > m\sqrt{n}$ if F_t is effective. Here are some examples of special situations where better bounds are known:

(a) Given a positive integer r and $n = 4^r$ in characteristic 0, it follows from [Ev] that F_t is effective if and only if $(t+1)(t+2) > nm(m+1)$ (for m sufficiently large, this is just $t \geq m\sqrt{n} + (\sqrt{n} - 2)/2$).
(b) In characteristic 0 when $m \leq 12$ and $n > 9$, [CM] also proves F_t is effective if and only if $(t+1)(t+2) > nm(m+1)$.
(c) The algorithmic bound given in [R1] gives very good bounds, typically better than $m\sqrt{n}$, as long as m is not too big compared with n; however, for m sufficiently large, the bound in Corollary V.1.1 is better (see [H2]).
(d) The best overall bounds in characteristic 0 seem to be those of [HR]. Although they are asymptotically about the same as those given here in the sense that they do not seem to lead to better bounds on Seshadri constants, they do typically give better bounds on effectivity of F_t for any given m. In fact, along with [Ev],

[HR] gives the only bounds currently known which are sharp in certain cases in which m and n can simultaneously (but not independently) be arbitrarily large.

V.2. Ampleness and Regularity

Here we consider questions (b) and (c); i.e., what is the least t such that F_t is ample, or such that F_t is regular?

Section II already gives a bound for (b): if F_c is nef, then F_t is ample for all $t > c$. (This is because L' meets all curves positively except for E_i, $i \leq n$, but F_c meets each E_i positively.) Consequently we have:

Corollary V.2.1: *If $d > m/\varepsilon_n$, then $dL' - (E_1 + \cdots + E_n)$ is an ample* **Q**-*divisor.*

Now we consider question (c). As was done in [Xu2], duality and the usual vanishing theorems can be used to convert bounds on nefness or ampleness into bounds on regularity. This approach gives part (b) of the next result. (Since [N2] completely solves the regularity problem for $n \leq 9$, we need only consider $n > 9$.)

Corollary V.2.2: *Let $F_t = tL' - m(E_1 + \cdots + E_n)$, as usual, and recall $d^* = \lceil \sqrt{n} \rceil$. Assume $n > 9$.*

- *(a) If $t \geq md^* + \lceil (d^* - 3)/2 \rceil$, then F_t is regular. If n is a square and $m > (d^* - 2)/4$, then the converse is true (i.e., the bound is sharp).*
- *(b) If $t \geq (m+1)/\varepsilon_n - 3$ but n is not a square, then F_t is regular.*

Proof: Part (a) follows from Lemma 5.3 of [HHF]. For part (b) let $H_{t+3} = F_t - K_Y$; i.e., $H_{t+3} = (t+3)L' - (m+1)(E_1 + \cdots + E_n)$. Then H_{t+3} is nef and big (i.e., $H_{t+3}^2 > 0$) by Corollary II.3 for $t + 3 \geq (m+1)/\varepsilon_n$, hence Ramanujam vanishing (see [Ra], or, in positive characteristic, Theorem 1.6 of [T]) implies $-H_{t+3} = K_Y - F_t$ is regular, so by duality $K_Y - (K_Y - F_t)$ is regular. $\diamondsuit$

The bounds given by this corollary seem to be the best general bounds now known, but in special cases better ones are known. For example, if $n > 9$ is a square but m is not too big, the bound in Corollary V.2.2(a) is known not to be optimal; in characteristic 0, [Ev] gives an optimal bound for all m if n is a power of 4. If m is not too big compared with n, the algorithmic bounds in [R2], although they are hard to compute, are often the best available (but for m sufficiently large, the bounds given here are better; see [H3]). In [HR], bounds are given in characteristic 0 which are better than and sometimes harder to compute but asymptotically about the same as those of Corollary V.2.2(b). The bounds of [HR] are, however, sharp for certain values of m and n which can simultaneously (but not independently) be arbitrarily large. Other bounds have also been given. Those of [Gi], [Hi] and [Ca] are on the order of $m\sqrt{2n}$, while those given here are on the order of $m\sqrt{n}$. Similarly, Corollary V.2.2 is better than the bound of [Bal] if m is large enough, and better than [Xu2] (Theorem 3) if n is large enough. See [H3] for a discussion and some comparisons.

V.3. Freeness and Very Ampleness

We now consider the last two questions, what is the least t such that $|F_t|$ is base point

free, and what is the least t such that F_t is very ample? The results of Section V.2 have an immediate application here. Indeed, it is well known that F_t is base point free as long as F_{t-1} is regular, and very ample as long as F_{t-1} is free and regular. (This follows from the fact that the ideal I_Z of the fat point subscheme $Z = m(p_1 + \cdots + p_n)$ is generated in degrees $t \leq \sigma$ [DGM], where σ can be defined as one more than the least t such that F_t is regular.) Thus, if F_t is regular for all $t \geq N$ for some N, then $|F_t|$ has no base points for $t \geq N+1$ and is very ample for $t \geq N+2$.

In certain cases, one can do better. For example, [HHF] shows that when $n > 9$ is an even square and $m > (\sqrt{n}-2)/4$, then F_t is both regular for all $t \geq m\sqrt{n} + (\sqrt{n}-2)/2$ and that I_Z is generated in degrees at most $m\sqrt{n}+(\sqrt{n}-2)/2$. Thus F_t is also base point free for all $t \geq m\sqrt{n}+(\sqrt{n}-2)/2$, and very ample for all $t \geq m\sqrt{n}+\sqrt{n}/2$. For additional (but characteristic 0) examples, when n is not a square but both n and m can be large, see [HR].

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